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SUNSET SCIENCE. IV. LOW-ALTITUDE REFRACTION

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ABSTRACT

Established authorities have disagreed about the relative importance of the upper and lower atmosphere in producing astronomical refraction for nearly two centuries. This paper resolves the problem and corrects some prominent errors. The refraction near the horizon is explored in some detail, and its relation to terrestrial refraction, and the effects of thermal inversions in the nocturnal boundary layer, are examined. At many observatories, the refraction at the apparent horizon comes mostly from the air between the observatory and sea level.
Key words: atmospheric effects

1. INTRODUCTION

This is the fourth in a series of papers (Young, Kattawar, & Parviainen 1997; Young & Kattawar 1998; Young 2000) on low-Sun phenomena. During this work, it has become apparent that textbook presentations of refraction emphasize the calculation of numbers, rather than insight—especially near the horizon. This paper is intended to rectify that lack.

For centuries, astronomers have known that atmospheric refraction out to about 75° zenith distance (Z.D.) depends almost entirely on the temperature and pressure at the observer, and not on the detailed structure of the atmosphere. This result was first proved by Oriani (1787a, 1787b), though the fact had already been noticed by Flamsteed, Newton, and other early workers. Oriani's theorem (as it is sometimes called) accounts for the success of Cassini's homogeneous-atmosphere model in this part of the sky; an elegant proof of the theorem in modern notation is given on pages 122–124 of Ball (1915), but without mentioning Oriani.

However, it was clear that the refraction near the horizon is *not* determined solely by the local temperature and pressure. Delambre (1814, pp. 319–320) pointed out that "in the vicinity of the horizon . . . from one day to another, and in circumstances that were apparently the same, the refraction varied by 15″ to 20″ without one being able to suspect the cause; but the variations are still more appreciable at the horizon."[1] Delambre offered several examples, including two days with the same temperature and pressure but with horizontal refractions differing by some 4′.

Indeed, much larger variations than these have been observed occasionally, particularly at high latitudes, beginning with the famous observations of the Dutch explorers led by Willem Barents, in 1597. They observed the first sunrise in spring 2 weeks earlier than expected, corresponding to a horizontal refraction of over 4°. The *typical* horizontal refraction near Hudson's Bay was found to be "more than a degree" by Captain Middleton (see Coats 1852, p. 132) in the winter of 1741–1742, a result confirmed by James Isham (see Rich & Johnson 1949, p. 73) in the same area the next year. In modern times, Nansen (1897) observed the Sun when it was 2°11′ below the horizon, and Shackleton (1962) reported a refraction of 2°37′. Recently, Sampson et al. (2003) reported refraction exceeding 2° at Edmonton, Alberta.

But large refractions are not confined to high latitudes: Bouris (1859) reported "whole series of stars regularly observed with the meridian circle that culminate at Athens up to 4° below the horizon, such as ξ Lupi, ϵ^1 Arae, ϵ^2 Arae, β Arae, . . . and Canopus," and I myself have observed sunsets delayed by more than 5 minutes in San Diego, California, corresponding to refraction more than a degree greater than normal.

The obvious distortions of the Sun at the apparent horizon have also puzzled many observers. The earliest systematic observations seem to be those of Le Gentil (1779, pp. 393–415), though scattered reports extend back to antiquity. Besides the references included in O'Connell's well-known book (O'Connell 1958), the visual observations of Beuer (1901), Schnippel (1901), Zona (1902), Graff (1906), Krčmář (1906), Doss (1907), Wetekamp (1908), Fisher (1920, 1921), Hurand (1930), Owen (1934), Moss (1938), Dines (1942), and Lamb (1947) may be mentioned. These distortions, first photographed by Colton (1895a, 1895b), by Riccò (1901), and by Rudaux (1906), have frustrated attempts (e.g., Hellerich 1928; de Kort 1960; Győri 1993) to use sunset photography for determining the astronomical refraction for correcting routine astrometric observations higher in the sky. Additional photographs were published by Rudaux (1927), by Chappell (1933), and of course by O'Connell (1958). Photographs continue to appear in Sky & Telescope (e.g., Baumgardt 1986; Sinnott 1986; Parviainen & Coombs 1987; Sampson 1993; Sinnott 1994; Brings 1995) and elsewhere.

2. NEGLECT

The attitude of astronomers toward low-altitude refraction has generally been that expressed by Brinkley (1815), who wrote, "It is well known to those conversant in observations made with good instruments that near the horizon an irregularity in refraction hitherto unexplained shews itself. This commencing even at less zenith distances than 80°, is at first very small, but increases to a very considerable irregularity as we approach the horizon" (p. 108). But, rather than understanding the problem, he decided that "the quantity of refraction varies so much from some unexplained cause, the heights of the barometer and thermometer remaining the same, that observations below 80° can be of little use" (p. 81). So he

[1] Unless otherwise noted, all translations are my own.

concluded that "it is not likely the irregularities will ever be submitted to any law, and investigations respecting formulae for refractions for zenith distances greater than about 80° may be considered more curious than useful" (p. 117). The problem has therefore remained largely ignored by both observers and theoreticians, despite the large body of unexplained observational material.

However, Brinkley's apathy should not stand unchallenged. A more productive view was expressed by F. W. Bessel (1823). In discussing his own observations below 5° altitude, he wrote, "In my efforts to obtain this material, I have regarded the refractions not only as a means of reducing the observed zenith distances, but believed that their determination possesses an interest in itself. Taken in this measure, the solution of the problem requires the extension of the tables all the way to the horizon; but if it were just a matter of the reduction of observations, the tables would only be necessary to those heights at which the stars still appear steady and clear; then the next 5° at the horizon could well remain completely unknown."

2.1. *Practical Applications*

Indeed, a thorough understanding of refraction near the horizon can improve reductions of observations in the part of the sky where astrometry is usually done, for if the low-altitude refraction is not understood, it is not clear how far standard refraction tables and formulae can be trusted. Different workers have proposed different limits between 75° and 85° Z.D. And if the variable lower atmosphere is important, local circumstances may determine the cutoff; what is reliable at one observatory might be unpredictable at another.

Furthermore, because data obtained from the Global Positioning System (GPS) satellites must be corrected for atmospheric delays, and the satellites are at low angular altitudes most of the time, the need to understand atmospheric effects near the horizon has recently become more important. According to Mendes et al. (2002), atmospheric refraction is the main limitation to geodetic accuracy in GPS, very long baseline interferometry, and satellite laser ranging.

2.2. *A Remark on Terminology*

Bessel's use of "height" (*Höhe*) in the passage quoted above raises a problem of terminology: just as in German and French a single term is commonly used for both linear height above sea level and angular distance above the horizon, the terms *altitude* and *height* are occasionally used in English in both senses. A referee has suggested that "zenith distance" would avoid the problem, but as I have pointed out elsewhere (Kasten & Young 1989), this leaves us with an ambiguity of notation when the structure of the atmosphere is involved: astronomers use z for zenith distance, but meteorologists always use z for the vertical linear coordinate. Besides, when discussing refraction near the horizon, the natural reference point is the horizon, not the zenith; so altitude, not zenith distance, is the more appropriate angle.

Such questions have been debated for years: Horton (1922) proposed "elevation" for height above sea level—a usage that has become standard in geography and surveying—but "altitude" for height above the ground, which conflicts with astronomical practice. Besides, "elevation" is sometimes used for angular altitude.

As concluded in Kasten & Young (1989), no compromise can satisfy everyone. I shall use *height* for linear distance above sea level, and *altitude* for angular distance above the horizon. But, occasionally, quotations from other authors will require the reader to be careful.

3. CONFLICTING EXPLANATIONS

3.1. *Proponents of the Upper Atmosphere*

Although the peculiar behavior of refraction near the horizon has not been thoroughly investigated, many writers have offered opinions about it. Some have stated that the cause lies in the upper atmosphere. For example, James Ivory, in his influential paper of 1823, says that the variations noted by Delambre and by Brinkley at and beyond 75° Z.D. "are undoubtedly produced by alterations in the remote parts of the atmosphere, which do not affect the barometer or the thermometer placed at the Observatory" (Ivory 1823, p. 432). No less an authority than Simon Newcomb (1906, p. 183) says that "astronomical refraction is little influenced by the diminution of temperature at low altitudes, the effect of differences of temperature reaching their maximum near the pressure-height, and slowly diminishing for yet greater heights. We must, therefore, for astronomical purposes, lay more stress on the temperature at considerable heights than near the surface of the earth."

Garfinkel (1944) suggested that "it can reasonably be expected that the theory will be corroborated further when the refraction data for great heights become more abundant and more reliable." Similarly, Woolard & Clemence (1966, pp. 85–86) say that "with the accumulated meteorological data on the upper atmosphere now available, theories of refraction may be based on empirical density distributions . . . and analytical developments that are theoretically valid at all zenith distances may also be used."

Furthermore, O'Connell (1958, p. 21) says, "One point emerges very clearly from our observations—the rim of the low sun can be very strongly disturbed by scintillation even when the distant horizon is extremely clear and sharply defined, showing that the cause of the scintillation is not always in the lower layers of the atmosphere." (The classic example is shown in his Plate 19, in which a ship on the horizon, 80 km away, is sharply silhouetted in front of the highly distorted image of the Sun.) He then quotes a passage from Wegener's (1928) encyclopedia article, in which the cause of scintillation is attributed to "the high and highest layers of the atmosphere." On the next page, O'Connell says, "our observations prove, I think conclusively, that strong scintillation can be observed near the horizon which in no way depends on conditions in the lower atmosphere." In his follow-up article in Endeavour, O'Connell (1961) concludes that "it is worth emphasizing that the appearance of the green flash, and of other phenomena referred to here, is dependent on . . . conditions at great altitudes." He mentions investigations of "these very tenuous upper regions of the air" by rockets and satellites and suggests that "studies of such phenomena as the green flash may contribute useful information" about them.

Other modern workers have also mentioned the importance of the upper atmosphere for refraction near the horizon. For example, Kolchinskii (1967, p. 14) noticed that refraction tables computed by numerical integration for different realistic model atmospheres differ much more at the horizon than would be expected if the refraction were proportional to the refractivity at the observer and concluded, "Thus, the dispersion between the refraction at the horizon, calculated by [refractivity scaling] and obtained by numerical integration, proves to be highly significant. It indicates that in this case the upper layers of the Earth's atmosphere influence the value of the refraction integral.''

In a widely cited paper, Saastamoinen (1972a) argues that the diurnally varying atmospheric boundary layer can be neglected, saying that ''since only a small contribution to these integrals comes from the lower levels, it will be quite sufficient merely to extend the constant temperature gradient of the free troposphere down to the ground level, neglecting the small error thus involved.'' In the continuation of this work (Saastamoinen 1972b), he adds that ''any significant variation of the tropospheric contribution, whether of regional or seasonal origin, is largely compensated by the stratosphere.'' More recently, Mikhailov (1975) has stated that ''beyond 60° . . . the state of the atmosphere at heights of 10 km and greater begins to exert an increasing influence'' on the refraction, and Yan (1996) has stated that a tropopause height different from that used in the standard model can ''produce errors of astronomical refraction.''

One could hardly ask for more definite statements than those quoted above.

3.2. *Proponents of the Lower Atmosphere*

On the other hand, there are equally definite statements that the lapse rate in the *lower* atmosphere is the important factor. Perhaps the first astronomer to express such an opinion was Bessel (1823), who attributed the dispersion of the observations at low altitudes to ''unknown irregularities in the lower layers of the air'' and pointed out that the repeated measurements of thermal structure made by Zumstein on Monte Rosa gave ''widely deviating results, and thus show that the lapse rate of the atmosphere is no less uniform;—one can add that the direct observations of the thermal gradient are always made in the daytime, those of the refraction at night, and that the heating of the ground surface by sunshine has apparently increased the lapse rate.—In my view one must find the lapse rate directly from the refraction observations . . . ; it seems probable that refractions observed by day would give a greater lapse rate than night observations.'' He also declared that this was substantiated by Argelander's observations of sunsets, which gave ''generally smaller refractions than those mentioned above, based on the fixed stars.''

Soon afterward, Atkinson (1830, 1831) attempted to show ''that the fluctuations in the state of the atmosphere near the surface of the earth are not only fully adequate to account for the very great variations which have been observed in the horizontal refraction, but even for still greater variations.'' But Atkinson died before completing his work.

A few years later, Henderson (1838) reported his observations of refraction near the horizon made at the Cape of Good Hope, showing day-to-day variations of a few tens of seconds near 85° Z.D., and variations exceeding a minute closer to the horizon. He remarked that beyond 88°, ''it is well known that the astronomical refractions are extremely irregular, being affected by the same causes which make the terrestrial refractions so variable and uncertain.''

Henderson did not say what these causes are; but in fact they had already been well established. Gruber (1786) had first demonstrated experimentally that mirages were produced by heated surfaces. Woltman (1800) made careful observations of mirages and terrestrial refraction and established that the temperature difference between the air at eye level and the surface below it was the decisive factor: ''always, if the water was about 2° Fahrenh. or more *warmer* than the air, a *depression* of the rays that extended over the water surface took place, and (assuming that the objects were visible) an inferior mirage. On the other hand, if the water was about 2° *colder* than the air, *raising* of the rays took place, and never an inferior mirage.'' Heinrich Wilhelm Brandes (1807) published a monograph on mirages and refraction in which he tabulated thousands of observations of terrestrial refraction, together with temperature differences measured at different heights up to $16\frac{1}{2}$ feet (5 m) above the ground. In a summary paper, Brandes (1810) stated as his first result that ''if one frequently observes the apparent height of individual objects above the Earth, and simultaneously investigates the heat of the air each time at different heights, one finds quite generally that the apparent height of each object is the greater, the warmer the higher layers of the air are in comparison with the lower ones.'' In the same year, J.-B. Biot (1810) published his own monograph on horizontal refraction and mirages, in which the theory was worked out in detail and extensive quantitative comparisons between measured altitudes and temperature gradients were used to confirm it. Biot's monograph is exceedingly thorough; it encompasses refraction, dip of the horizon, superior as well as inferior mirages, looming, etc. So the importance of temperature gradients in the very lowest part of the atmosphere was already well established early in the 19th century.

Although many review articles on mirages and refraction were subsequently published, most of them appeared in the meteorological or geodetic literature and may have been overlooked by astronomers. However, in 1836 Biot published another very extensive treatment of refraction in the Additions a la Connaissance des Tems pour l'An 1839, which should have made the astronomical world well aware of the subject. In this, he emphasized how the local zenith angle rapidly departs from 90° in the upper atmosphere, even for rays that are nearly horizontal at the observer. This effect rapidly diminishes the relative contribution of the upper layers.

Similar arguments were made by Fabritius (1878), who pointed out that the refractive invariant along the ray, namely, $nR \sin \zeta = n_o R_o \sin z$ (in which ζ is the local zenith distance of the ray at a distance R from the center of Earth, where the local refractive index is n, and the corresponding terms on the right-hand side refer to the observer's position), ''shows that the incidence angle . . . will differ very appreciably from z as soon as $\sin z$ is near unity.'' Consequently, even at the observer's horizon, the local zenith distance of the ray at the height where nR exceeds its value at the surface by only 1% is only about 82°, so that ''one may draw the conclusion that the constitution of the atmosphere in the highest layers would be without any appreciable influence even on the horizontal refraction.''

Likewise, in a detailed observational study of terrestrial refraction, Heinrich Hartl (1881) concluded, ''On the grounds of purely empirical researches, influenced by no hypotheses, I believe to be justified in stating: *The temperature decrease with height is the most important factor in the daily and yearly periods of terrestrial refraction*; the other meteorological elements are of only secondary importance.'' Inasmuch as the terrestrial refraction is simply the contribution of the lowest layers in the atmosphere to the astronomical refraction near the horizon, the variations in astronomical refraction at low altitudes must largely be due to these layers.

Fletcher (1952), in a fine review of low-altitude refraction, stated explicitly that ''at low angular altitudes the lowest layers in the atmosphere, the lowest few kilometres, influence the refraction much more than at higher altitudes; the temperature gradient near the ground is known to vary systematically with season, time of day, type of weather and so on, and also to vary in a random or accidental manner.'' He then devoted a whole two-page section to the "Influence of Temperature Gradient near the Earth's Surface," with numerical examples, calling it "perhaps the chief systematic cause of variability of refraction at very low altitudes." Recently, van der Werf (2003) has found similar results.

One could hardly ask for more definite statements than these. But they are completely incompatible with the assertions cited in the previous section. So whom should we believe, Ivory, Newcomb, Garfinkel, Woolard & Clemence, and O'Connell—all distinguished astronomers—or Bessel, Henderson, Biot, et al.? It is remarkable that the conflict between their statements has never been noticed, much less resolved.

4. RESOLVING THE DILEMMA

4.1. *The Open Question*

To begin with, I must point out that those who attribute the variations in near-horizontal refraction to the temperature gradient in the lower atmosphere have offered more quantitative arguments than those who favor the upper atmosphere. However, even these writers have not shown *how much* of the lower atmosphere is important: is it the whole troposphere, or only some restricted region near the surface? This part of the question was left open, so that Mahan (1962), in reviewing the whole subject of astronomical refraction, wrote that "the importance of the stratosphere is . . . still an unsolved one."

A simple way to settle the issue would be to find the height in the atmosphere that divides the total refraction at each given zenith distance into two equal parts. This requires a method of calculating refraction that preserves the contribution of each elemental layer into which the atmospheric model is divided. The method must be valid all the way to the horizon—and, as we shall see, even beyond it.

4.2. *The Method of Solution*

What is needed is a detailed quantitative accounting of where in the atmosphere the astronomical refraction near the horizon comes from. In the days of analytical theories, this was not easy to do; the definite integrals used often had infinite upper limits, and breaking the integration within the atmosphere was not easily done, not least because of the transformations of variable required to make a solution possible at all. And a straightforward numerical integration of the untransformed expressions was exceedingly laborious.

However, all this should have been changed by Biot's 1836 memoir on astronomical refraction. In it, he introduced a simple transformation that allowed him to calculate the refraction for an arbitrary atmospheric model with only a 13-node quadrature. Unfortunately, Biot was too far ahead of his time: numerical methods would not come into favor until the advent of computers. His method was forgotten, to be rediscovered, a century and a half later, by Auer & Standish (1979, 2000). It is now the recommended method (Seidelmann 1992) for calculating refraction.

Unlike the traditional semiconvergent series approximations introduced by Lambert (1759), the Biot-Auer-Standish (BAS) transformation is exact: it merely changes the variable of integration to the local zenith distance at each point along the ray. This removes the divergence of the integrand at the horizon that is so troublesome in the standard form involving tan z.

It should be remembered that the BAS method fails when the ray curvature approaches that of Earth. This means it is not applicable to cases of extreme refraction, such as those that involve ducting, or the Novaya Zemlya phenomenon. However, we can settle the question adequately without considering such cases.

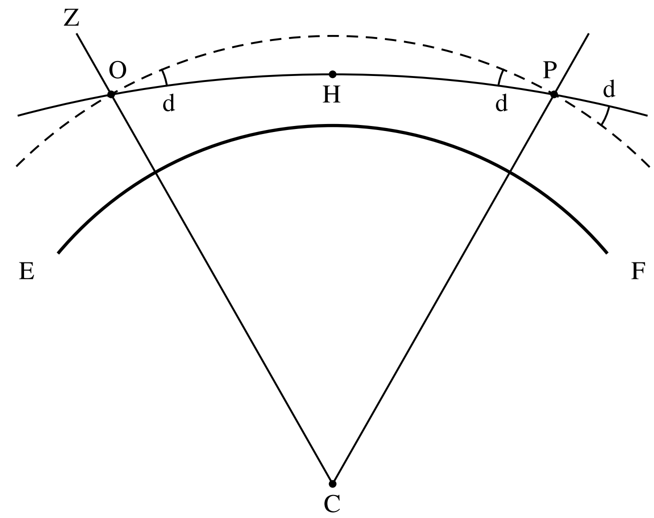

FIG. 1.—Use of symmetry to economize refraction calculations below the horizon. Solid arc EF represents Earth, with center at C. The observer is at O, looking along the ray OHPS, which is horizontal at H, its lowest point. The dashed arc is the horizontal surface through the observer; the dip of the ray at the observer is *d*. Note the equality of the three angles marked *d*.

4.3. *A Few Details*

The BAS method divides the atmosphere into layers and calculates the contribution of each layer separately to the total refraction, so we can just save the partial sums layer by layer and compare them with the total. To determine the exact half-contribution point, I have subdivided the original layers into sublayers, each of which is smaller than 1/256 of the total thickness of the original layer. This provides values at the intermediate tabulated points, so the midrefraction point can be found by linear interpolation.

To calculate refraction at a dip *d* below the astronomical horizon, determine the lowest point ("H" in Fig. 1) on the ray from the refractive invariant. The ray is symmetric about H. As the refraction from O to H is the same as that from H to P, twice the refraction from H to P is the refraction from O to P—that is, the contribution to the required refraction from the air *below* the observer. The contribution from the air *above* the observer is simply the refraction calculated for an altitude of *d* as seen from P, at the same height as the observer. The sum of the refractions from the air above and below eye level then gives the required result. This is equivalent to integrating the horizontal refraction from H, but doubling the accumulated refraction when the point P (at the observer's height) is reached.

The program used is based on the one published by Hohenkerk & Sinclair (1985), but with much tighter convergence limits in the iterations, and error limits changed from absolute (seconds of arc) to relative (fractional) tolerances of 1 part in 10^8. A more accurate formula for the refractivity of dry air (Peck & Reeder 1972) is also used. The calculations here have been done for a wavelength of 700 nm, which is appropriate for observations very near the horizon.

5. EXAMPLES

5.1. *The Standard Atmosphere*

The US Standard Atmosphere (Committee on Extensions to the Standard Atmosphere 1976) is a widely used reference

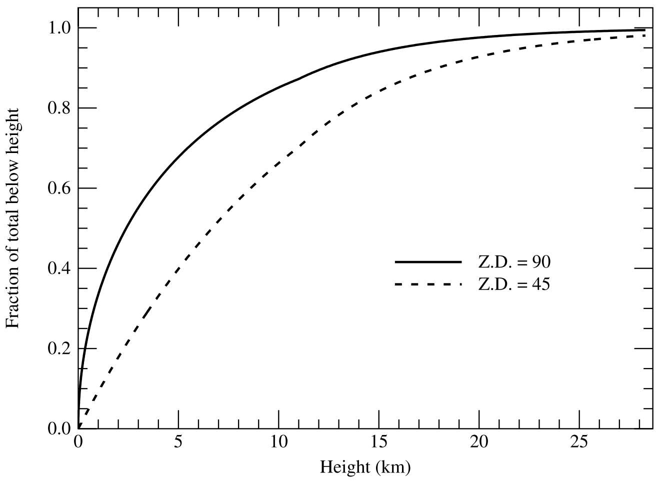

FIG. 2.—Fraction of the astronomical refraction that arises below different heights in the US Standard Atmosphere, for an observer at sea level. *Solid line*, horizontal refraction; *dashed line*, refraction at 45° Z.D.

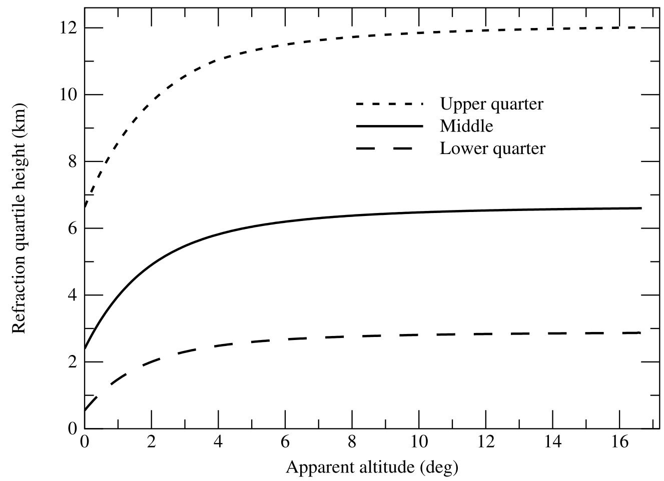

FIG. 3.—Heights of the quartile contributions to astronomical refraction in the US Standard Atmosphere as functions of altitude, for an observer at sea level.

model; it is very similar to the International Civil Aviation Organization model atmosphere that preceded it, and even to models used in the 19th century, because it has been known for a long time that the mean lapse rate in the troposphere is near 6 K km^{-1} (cf. Ivory 1823). For our purposes, the important values are the temperature and pressure at the surface, and the tropospheric lapse rate; for the Standard Atmosphere, these are 15°C, 1013.25 hPa, and 6.5 K km^{-1}. This lapse rate, which Newcomb (1906, p. 183) called "the general average, day and night, taking the whole year round," extends up to 9 km in the Standard Atmosphere, whose isothermal stratosphere begins at 12 km.

As one would expect, the vertical distribution of the contributions to the refraction is nearly the same for all zenith distances less than 45° or so, where a plane-parallel model is nearly correct. Half the refraction comes from below a height of 6671 m at 30° Z.D.; 6666 m at 45°; and 6653 m at 60°. The first and third quartiles turn out to be about 2.9 and 12.1 km, respectively; the dashed line in Figure 2 shows the fraction of the total below each height at 45° Z.D. The weak dependence on zenith distance is not surprising.

The actual value is not very unexpected, either: as the ray bending is proportional to the density gradient, the mid-refraction point should correspond roughly to the place where the density has half its value at the surface, and this is very nearly true. A little more than one-quarter of the near-zenith refraction comes from the stratosphere. Similar results were obtained by Emden (1923).

However, the numbers are very different near the horizon. At the horizon itself, half the refraction comes from below 2395 m height—a region that contributes less than a quarter of the total refraction at small zenith distances. Three-fourths of the total refraction at the horizon comes from the air below 6633 m height. That is, the region above this height contributes only a quarter of the refraction at the horizon, while it contributed very nearly half of the total near the zenith. The region above 12.1 km, which produces a fourth of the total refraction near the zenith, contributes only 10% of the total at the horizon (see the solid line in Fig. 2). These numbers support Fabritius's (1878) views. More remarkably, at the horizon, a quarter of the total refraction comes from just the lowest 545 m of air. The increased importance of the *lower* atmosphere at the horizon is obvious.

The changes near the horizon are very rapid, as Figure 3 shows. Above 10° altitude, the refraction contributions are little different from the values near the zenith. Nearly all the change occurs below 5° altitude.

5.2. *Elevation of the Eye*

The values just given apply to an observer at sea level, but no one ever observes from the very surface of the sea. Even standing at the shoreline, a person of average height observes from about 1.5 m above the sea, at which height the dip of the horizon is over 2′. For such an observer in the Standard Atmosphere, half the refraction at the *apparent* horizon comes from the bottom 2323 m, and 2.6% of the total refraction comes from the 1.5 m of air *below* eye level.

It may seem surprising that the lowest 1.5 m of air—less than 0.02% of the whole atmosphere—could contribute 2.6% of the total refraction at the apparent horizon. The explanation is that the ray touching the apparent horizon has a very long curved path in these 1.5 m of height: about 9.5 km, half on either side of the horizon. The relative air mass at the horizon is about 38, corresponding to a total path length equivalent to some 300 km of air at sea-level density, so the path below 1.5 m height is about 1/32 of the total. Thus, it should not be surprising that this bottom layer contributes nearly 3% of the total refraction.

One-quarter of the refraction at the sea horizon comes from below eye level for an observer 175 m above sea level, and half the total comes from below eye level for an observer at 992 m. It turns out that the percentage of refraction at the apparent horizon that is produced below eye level is nearly equal to the dip of the horizon in minutes of arc, for the Standard Atmosphere (see Fig. 4). For example, at the Vatican Observatory (450 m above the sea), where O'Connell's observations were made, the dip of the sea horizon is 37.′4, and 37.1% of the refraction at the sea horizon comes from the air below the observatory.

Colton (1895a, 1895b) and Chappell (1933) observed sunsets at Lick Observatory, about 1290 m above sea level. About 55% of the refraction produced by the Standard Atmosphere at the Lick sea horizon comes from the atmosphere below the observatory. For the typical modern observer at a mountain observatory near 2000 m elevation, over 63% of the astronomical refraction at the sea horizon comes from below eye level. At the Carlsberg Meridian Telescope on La Palma, at 2326 m, over 66% of the astronomical refraction at the sea

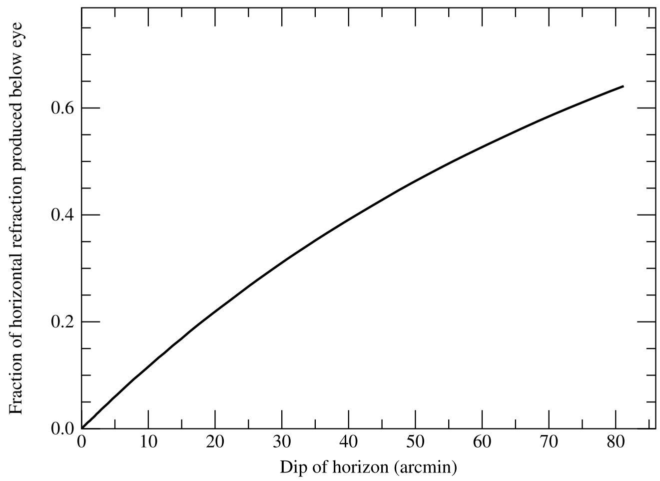

FIG. 4.—Fraction of refraction at the apparent horizon produced by the air below eye level, as a function of dip of the horizon, for the US Standard Atmosphere.

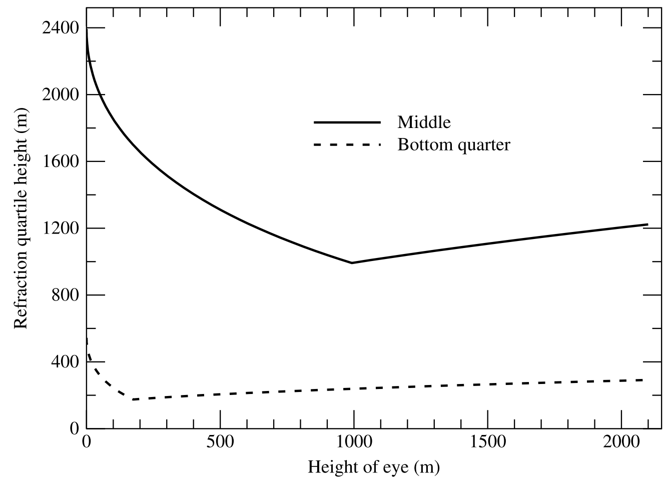

FIG. 5.—Heights of the lower two quartile contributions to refraction at the visible horizon in the US Standard Atmosphere, as functions of height of the observer's eye.

horizon is due to the air below the telescope. Evidently, the refraction at the visible horizon seen by astronomers at mountain observatories comes mainly from the air beneath their feet.

Figure 5 shows the effect of eye height for the lower two quartiles of the refraction at the apparent horizon; the upper quartile behaves similarly, but it cannot be shown clearly on the same scale. As the observer's height above sea level increases, the region below the eye (in which the air path is doubled, because of the part between the eye and the horizon) also increases. As long as this region is below the height of the given quartile, its double contribution serves to reduce the quartile height as the eye height increases, so the curves initially slope downward. However, this effect diminishes with increasing eye height, because the biggest contribution comes from the lowest levels, where the ray is most nearly horizontal. Consequently, the slope of the curve decreases as the eye height increases.

When the height of the eye coincides with the quartile height, the latter reaches a minimum. Thereafter, as the total refraction at the apparent horizon increases with eye height, so must the quartile height. So there is a discontinuity of slope at the minimum of each curve.

6. EFFECTS OF THERMAL INVERSIONS

6.1. *The Boundary Layer*

Users of the Standard Atmosphere should recognize that such models are extremely unrealistic: the real atmosphere is never in such a state. This model averages out diurnal variations, which are particularly large over land. In the real atmosphere, daytime solar heating and nocturnal radiative cooling occur mostly at the ground surface and gradually propagate upward into the overlying air. Soon after sunrise a convective boundary layer forms at the surface, gradually growing thicker at the expense of the stable layer formed in the previous night. Around sunset, radiative cooling of the ground usually forms a thermal inversion, and the stable boundary layer thickens during the night.

Thus there is *never* a time when the whole boundary layer has the constant temperature gradient of the Standard Atmosphere: some parts of the boundary layer are near the adiabatic lapse rate, while others have an inverted lapse rate. Thermal structure is always present in the lowest few hundred meters of air. Because most astronomical observations are made at night, under clear skies, a radiative thermal inversion is usually present at the surface.

Indeed, even Newcomb (1906, p. 183) recognized this fact, and he prefaced his assertion that refraction is "little influenced by the diminution of temperature at low altitudes" by remarking that this decrease "is changed to an actual increase"—that is, a thermal inversion—during the night, so that "were the rate of diminution near the surface of the earth important, it would be necessary to suppose a very small rate in the lowest kilometre of the air for the purpose of computing the astronomical refraction for night observations." As, in fact, the lapse rate near the surface *is* important, the investigation of a surface-based thermal inversion is necessary.

6.2. *Models with Inversions*

Nocturnal inversions in the boundary layer are quite varied, with depths usually between 100 and 500 m, and increases in potential temperature typically between 5 and 15 K (see chapter 12 of Stull 1988), though much stronger inversions sometimes occur, especially in winter at high latitudes. As examples, I shall consider inversions with a depth of 200 m and a deviation from the Standard Atmosphere of 10 K.

One can imagine several different ways of adding a thermal inversion to the Standard Atmosphere (see Fig. 6). The simplest is to keep the standard model above the inversion and simply cool the ground 10 K, as in model NBL1 in Figure 6. However, it is then difficult to disentangle the effect of the inversion from the change in refractivity at the observer.

But if we keep the surface temperature fixed, how should the atmosphere above the inversion be treated? One could raise *all* the temperatures above 200 m by the same amount, but the model then differs from the standard one everywhere but at the ground; then the problem is to distinguish between the effect of the inversion and the general heating of the atmosphere.

An intermediate approach is to bring the profile back to the standard one at some height above the inversion. The models NBL2–NBL7 return to the standard profile at heights of 1, 2, . . . , 6 km, respectively.

Figure 7 shows the resulting refractions near the horizon. Except for model NBL1, in which the surface temperature was

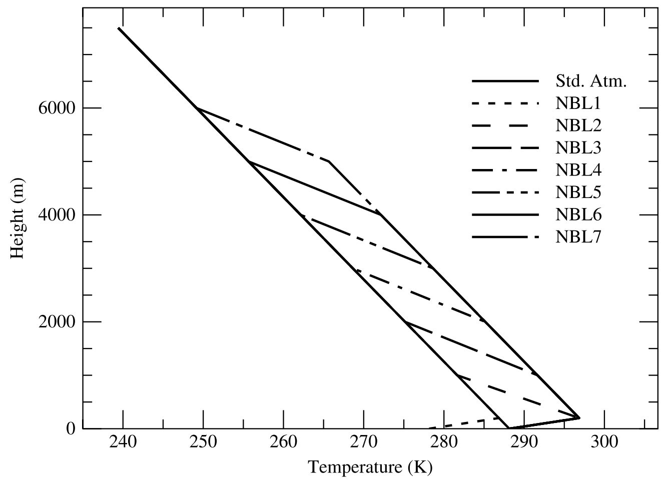

FIG. 6.—Atmosphere models with a 200 m deep thermal inversion of amplitude 10°C added to the Standard Atmosphere in various ways (see text for explanation).

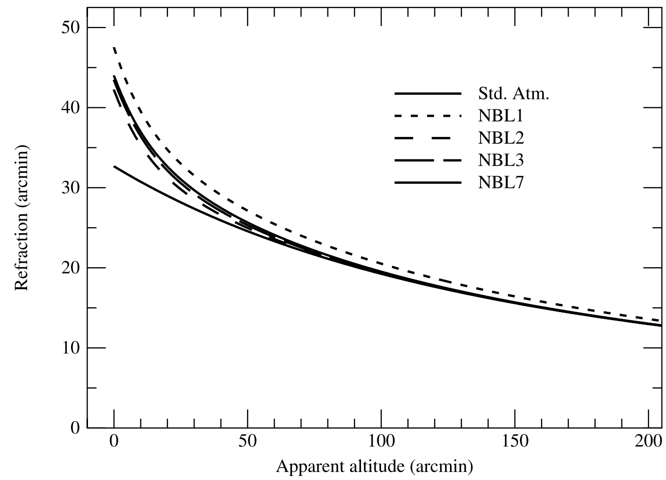

FIG. 7.—Refraction in the lowest 3°20′ above the horizon, for the models of Fig. 6, and an observer at sea level. At the scale of this figure, the curves for models NBL4–NBL6 are indistinguishable from the NBL7 curve and so are omitted; even the NBL3 curve is barely distinguishable from that for NBL7.

lowered, all the models have the same temperature and pressure at the observer, and they all produce nearly the same refraction as the Standard Atmosphere down to about 2° above the horizon. The region within 2° of the horizon that is affected by the nocturnal inversion depends on the thickness of the inversion layer; the fact that Ivory (1823), Henderson (1838), Strand (1952), and many other authors mention that the variations in low-altitude refraction are most pronounced in a zone of this width just reflects the fact that the nocturnal inversion is typically a few hundred meters thick.

As one would expect, the NBL1 refraction is about 3% larger than the other models, simply because of the corresponding increase in refractivity at the observer. However, apart from this difference in scaling, *all* the models with inversions behave very similarly: they all produce a much more rapid increase in refraction at the horizon than the Standard Atmosphere, and the differences among them are small compared with their common deviation from the standard refraction. Evidently the large increase in refraction at the horizon is the direct result of the inversion at the base of the nocturnal boundary layer. (Evidently, too, the effect of the inversion is confined to so small a zone of sky near the horizon that it would never be noticed in ordinary astrometric observations, which rarely extend beyond 80° Z.D.) The moderate nocturnal inversion increases the horizontal refraction by more than 10′.

The great similarity of the refraction curves for the models with inversions is a consequence of Biot's (1836) theorem that the altitude derivative of the refraction at the horizon depends on the temperature gradient at the observer. All the inversion models in Figure 6 have the same lapse rate at the surface, so of course their curves in Figure 7 are parallel at the horizon. Biot calls the slopes of these curves the "coefficient of variation" and says, "But just as, near the zenith, the coefficient of variation depends only on the refractive power observable in the layer where the observer is found, its value at the horizon depends at once on this power and on its immediate decrease as one rises above the observer, so that the action of distant layers has absolutely no effect."

Although the models NBL2 to NBL7 differ considerably above the inversion (cf. Fig. 6), their differences above 200 m produce relatively small differences in refraction. And as the jog in the profiles of Figure 6 rises from 1 to 6 km, its incremental effect on the refraction curve in Figure 7 rapidly diminishes. In fact, models NBL4–NBL7 are indistinguishable at the scale of Figure 7; their differences are only a few seconds of arc, even at the horizon. This shows how much more important the lowest layers of the atmosphere are for near-horizontal refraction than layers even a few kilometers higher up. (Similar conclusions were reached by van der Werf [2003], using a different family of thermal profiles.)

These facts completely contradict Newcomb's assertion that the sensitivity to thermal gradients increases from the ground up to about the pressure scale height. Likewise, O'Connell's assertions that the upper atmosphere is important are clearly wrong. Conversely, this simple example shows how strongly the refraction near the horizon can be affected by thermal structure in the lowest few hundred meters of air, while it is remarkably insensitive to the higher layers. To the evidence presented here may be added the fact (Young et al. 1997) that still weaker thermal inversions *below* eye level can produce extreme distortions of the low Sun, and even inverted (i.e., mirage-like) images.

6.3. *Depth of the Inversion*

Figures 8 and 9 show the effect of changing the inversion depth. In the models NBL8 and NBL9, the inversion depth is 400 m instead of the 200 m considered above. NBL9 has the same lapse rate as NBL7, so its double-thickness inversion makes most of the lower troposphere 20 K warmer than the standard model, instead of 10 K warmer. Figure 8 shows the profiles, and Figure 9 shows the resulting refractions near the horizon.

As we should expect from Biot's theorem, models NBL7 and NBL9, which have the same lapse rate at the observer, also have the same refraction gradient at the horizon. But although the inversion is twice as thick in NBL9, the horizontal refraction is only a little larger than for NBL7. That is, the second 200 m of inversion layer, from 200 to 400 m elevation, has produced very much less effect than the first 200 m above the observer.

The other striking feature of Figure 9 is that although NBL7 and NBL8 produce appreciably different refractions at the horizon, they are almost identical a degree or more above the

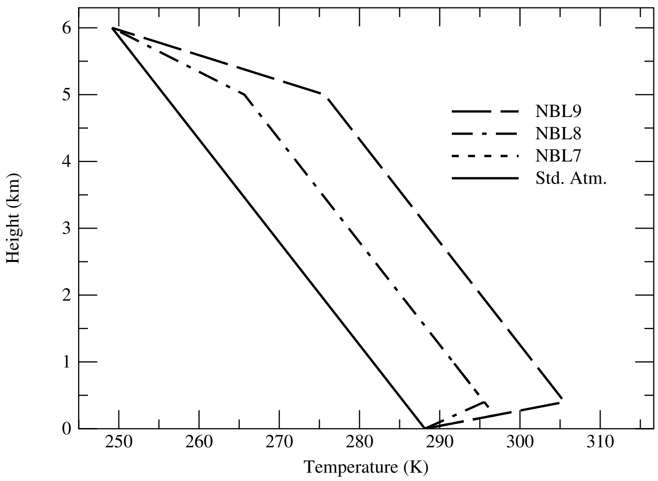

Fig. 8.—Models comparing three different inversions. NBL7 and NBL8 differ only below 400 m; NBL7 has twice the surface lapse rate, but its inversion is only half as deep. NBL9 has the same surface lapse rate as NBL7, but extending up to 400 m instead of 200 m. The corner is at 400 m in both NBL8 and NBL9. All models coincide with the Standard Atmosphere above 6 km.

horizon. That is, the bottom 200 m of air has affected only about the lowest degree of sky above the horizon.

6.4. *Elevated Inversions*

The similarity of the refractions for models with the same lapse rate at the observer but different thicknesses (NBL7 and NBL9), except near the horizon, suggests that the effects of elevated inversions will be small. A numerical experiment with a 10 K inversion between 200 and 400 m bears this out: except very near the horizon, it is essentially the same as the results shown for NBL7 and NBL8 in Figure 9.

However, as would be expected from Biot's theorem, it is closer to the standard case near the horizon, differing from it by 1′ at 45′ altitude, and 2′.83 at the astronomical horizon. Thus it would lie just above the solid line in Figure 9 and join NBL7 and NBL8 near their confluence about a degree above the horizon. To avoid cluttering the figure, it is not shown.

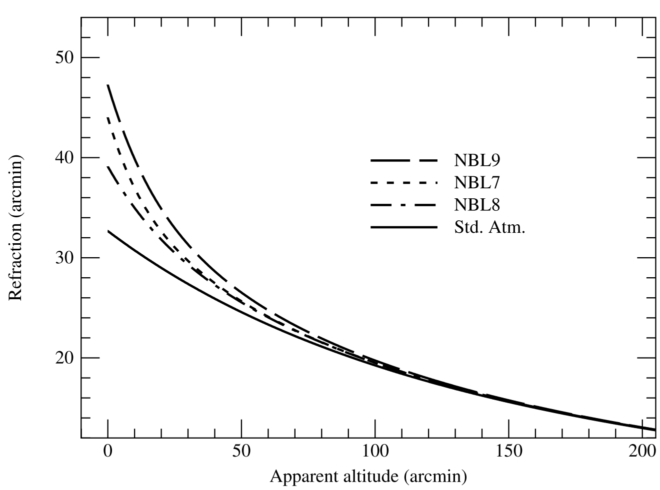

Fig. 9.—Refraction in the lowest 3°20′ above the horizon, for the models of Fig. 8 and an observer at sea level. Note that zero is off-scale on the ordinate axis. NBL7 and the Standard Atmosphere are also shown in Fig. 7.

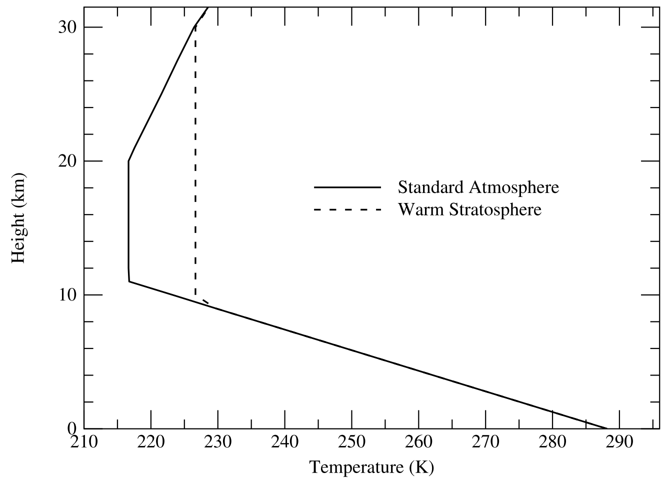

Fig. 10.—Temperature profile for a modified model atmosphere with 10 K warmer stratosphere.

7. THE STRATOSPHERE

7.1. *Changing the Stratosphere*

As several authors have suggested that the stratosphere plays an important part, it is useful to compare models having modified stratospheres with the Standard Atmosphere. For example, consider the warmer stratosphere shown in Figure 10, where the stratospheric minimum has been increased by 10 K. (To do this without changing the lower atmosphere, it is necessary to warm the entire region from 9 to 30 km, though only the heights from 12 to 20 km are a full 10 K warmer than in the standard model.)

As might have been expected from the theorems of Oriani and Biot, fixing the density and lapse rate in the lower atmosphere leaves very little room for variations in the refraction due to the upper atmosphere. The warm-stratosphere model increases the refraction by a milliarcsecond at 73°.5 Z.D., by a hundredth of a second at 80°.2, by a tenth of a second at 84°47′, and by a whole second of arc at 88°54′. The increase at the horizon is 1″.2. These small changes are probably all below the random errors of measurements at the corresponding altitudes and would be exceedingly difficult to detect.

A model constructed in the same way with a 10 K colder stratosphere, maintaining the standard lapse rates on either side of the minimum, changes a much smaller interval (from 12 to 19 km) and produces even smaller changes in refraction.

7.2. *Removing the Stratosphere*

As an extreme example of the unimportance of the stratosphere, so long as the conditions (including the lapse rate) at the observer are held fixed, consider a model with no stratosphere at all: we keep the lower layers identical to the Standard Atmosphere but hold the lapse rate fixed at 6.5 K km^{-1}. This model terminates at 44.33 km height. Even at the astronomical horizon, its refraction differs from that of the Standard Atmosphere by less than 2″.3.

For practical purposes, variations in stratospheric structure produce negligible variations in refraction.

7.3. *Explanation*

It seems paradoxical that although the stratosphere contributes about a quarter of the refraction near the zenith, major changes in it have little effect on refraction. Even at the horizon, it contributes more than an eighth of the whole refraction (see Fig. 2). So why is the low-altitude refraction so insensitive to the stratosphere's structure—or even its existence? The answer is basically that given by Biot (1836): circumstances conspire to make Oriani's result apply to *all* rays that reach the surface of Earth from the upper atmosphere, so the structure of the stratosphere is unimportant.

Consider first the angle at which the sea-level horizon ray meets the tropopause. The refractive invariant $nR \sin \zeta$ is simply nR where the ray is horizontal—that is, at sea level. If we assume the tropopause height is 10 km, $R_{\rm trop} = 1.00157 R_o$. The density at the tropopause is a third, or less, of the density at sea level, so $n_{\rm trop} = 1.0001$. As $n_o = 1.0003$, we see that $\sin \zeta = 0.9986$, corresponding very nearly to $\cos \zeta = 0.053$, or $\zeta = 87^\circ$. So the horizon ray has an altitude of about 3° at the tropopause. (This is the dip of the horizon as seen from a height of 10 km.)

But, as Figures 7 and 9 show, the refraction above 3° altitude is practically unaffected by structure above the observer, even at sea level. At 10 km, ray curvature is smaller than at sea level, so the effect of overlying structure above 3° altitude at the tropopause (corresponding to the horizon ray at sea level) must be still less. Any appreciable effects of stratospheric structure must be confined to rays closer to the tropopause observer's horizon than the sea-level horizon ray; such rays cannot reach sea level.

8. EXPLANATIONS OF ERRORS

As the true state of affairs has been stated repeatedly by numerous authors, from Biot (1836) onward, without stemming the flow of misinformation in the astronomical literature, merely asserting the truth once again may not suffice to eradicate the errors. Perhaps demonstrating the fallacies of the incorrect arguments will help.

Garfinkel (1944) and Woolard & Clemence (1966) are so brief about their claims that they offer no arguments to refute. But Newcomb (1906) and O'Connell (1958, 1961) do offer some substance, and Ivory (1823) makes remarks on both sides of the issue. The supposed connection made by Kolchinskii (1967) is nonsense, but the underlying error has been made by many others. Each of these last four authors deserves separate discussion.

8.1. *Ivory*

In addition to the remark already quoted, Ivory (1823) has a similarly wrong, or at least misleading, comment a few pages later (p. 436): "The refractions are ... affected by circumstances of which the observer has no intimation, and which cannot enter into any theory. The real causes of such anomalies is [*sic*] undoubtedly the irregular changes that take place in the remote parts of the atmosphere, which are not indicated by the barometer or the thermometer."

Yet, elsewhere he seems aware that the responsible parts are not so remote as these remarks would suggest. In particular, on page 424 he discusses the lapse rate (or rather, its reciprocal) and says that "this quantity is subject to great irregularities, which are not well understood. It is found that the refractions near the horizon are liable to variations equally irregular and unknown. There can be little doubt that both these effects are produced by the same causes, which disturb the gradation of heat, and the arrangement of the strata of air near the earth's surface." Here he clearly is correct, though somewhat lacking in conviction—especially compared with the "undoubtedly" he attaches to the "remote parts of the atmosphere" twice.

On page 472, Ivory again makes a stab in the right direction, but even more feebly: "With regard to altitudes less than 2°, it is not clear that the astronomical refractions do not participate of the extreme irregularity that attends the terrestrial refractions"—thus agreeing with Henderson's (1838) remark, but only by way of a double negative.

On page 456, Ivory discusses a variety of models, and finds that "although the refractions near the zenith are affected in a degree hardly perceptible by the peculiar constitution of the atmosphere,"—he seems not to have read Oriani (1787a, 1787b)—"yet, near the horizon, they depend entirely on the same arrangement of the strata of air indicated by terrestrial experiments." That is, the low-altitude refraction is satisfied only by atmospheric models that reproduce the observed lapse rate in the lower atmosphere, which Ivory takes to be 1°C in 90 or 95 English fathoms, corresponding to about 5.8–6.1 K km^{-1} in today's units. But then he immediately blames "the causes of the irregularities" on "the remote parts of the atmosphere" once again. Having had the correct explanation within his grasp, he let it escape.

8.2. *Newcomb*

The statements from Newcomb (1906) quoted above follow a comment that they are justified "for reasons which will be better understood when the general theory is developed," but as he never explicitly returns to the matter, it is not obvious what he had in mind. However, as Newcomb specifically excludes "an investigation of refraction near the horizon" from consideration (p. 223), he must have been thinking of the refraction in the part of the sky where astronomical observations are usually made. And, as his "General Investigation" section (p. 203) derives only the standard series expansion in powers of $\tan z$ or $\sec z$—and as, by Oriani's theorem, only the fifth-power and higher terms involve the detailed structure of the atmosphere—it must be these high-order terms that Newcomb had in mind.

To understand Newcomb's thinking, we must recall how this series expansion, originally due to Lambert (1759), is developed. Newcomb writes the refraction as

$$R = a \tan z \int_0^1 (1 + 2u \sec^2 z)^{-1/2} dw \tag{1}$$

(Newcomb's eq. [15a] on p. 208), where w is a dimensionless scaled density, z is the apparent zenith distance at the observer, and u is an unpleasant function that accounts for the difference of the ray curvature and Earth's curvature at each point along the ray. As these curvatures are both small, u is generally less than 0.01. So as long as $\sec^2 z < 50$, $2u \sec^2 z < 1$ and the square root can be expanded by the binomial theorem in a series in powers of $\sec z$. Then, reversing the order of summation and integration, so that the series can be integrated term by term, gives

$$R = a \tan z \,(1 - m_1 \sec^2 z + m_2 \sec^4 z - \cdots). \tag{2}$$

Each coefficient m_n is a definite integral involving the function u; apart from a numerical factor, this coefficient is essentially the $(n-1)$th moment of the density as a function of height. The signs alternate because of the $-\frac{1}{2}$ exponent in the original refraction integrand. The leading term is just 1; the coefficient m_1 of the next term involves the zeroth-order moment of the density distribution, which is simply the height of the homogeneous atmosphere; and only the higher order terms involve details of the density distribution. (The fact that m_1 depends only on the density of air at the observer is essentially Oriani's theorem.) Newcomb apparently had in mind the dependence of m_2 on the first moment of the density when he wrote his remark about the pressure height.

Newcomb says that the whole series is only convergent so long as sec z is less than about 7, corresponding to $z = 82°$. However, the convergence is actually slower than Newcomb thought: his value of 0.0000014 for m_2 calculated from Ivory's hypothesis is wrong. It should be 0.0000036, nearly 2.6 times larger than Newcomb's value. In Newcomb's development of Ivory's model, $m_2 = 18\nu^2/7 - 24\nu\alpha/11 + \alpha^2/2$, where $\nu = 0.00130$ is the ratio of the height of the uniform atmosphere to the radius of Earth and $\alpha = 0.000283$ is essentially the refractivity of air for standard conditions. If the numerical coefficients 18/7 and 24/11 of the first two terms are replaced by unity, Newcomb's value is reproduced exactly. It appears that he simply lost those coefficients in evaluating m_2—that is, he used just ν^2 instead of the whole ν^2 term, and similarly for $\nu\alpha$.

Indeed, the convergence is even worse than this, as is more obvious in the isothermal model developed in Newcomb's § 115 (pp. 214–215). In this case, Newcomb's equation (31) shows that the leading term of m_n is $n!\nu^n$, a sequence that must ultimately diverge, no matter how small ν is; worse yet, there is an additional factor (denoted by Newcomb on p. 211 as [i], for the ith term) that also grows in a factorial manner. In short, the series as written is only semiconvergent and must ultimately diverge. (This problem is discussed by Ivory [1823] on p. 467 of his paper.) This shows the danger of trying to infer the behavior of the whole refraction from that of the individual terms in this series.

Still, one might explain the height dependence of the refraction in terms of the series expansion. Because successive terms involve higher moments of the density distribution, higher order terms involve structure higher in the atmosphere. (This fact is often concealed by the numerous transformations of variable introduced in developing and integrating the series expansion; it is most evident in the treatment by Saastamoinen [1972a].) Then the alternation of signs in the series means that each term *reduces* the contribution to refraction from the upper part of the region emphasized by the previous term. So as sec z increases, the increasing relative size of the higher order terms progressively reduces the importance of the upper atmosphere, and the main source of the refraction steadily moves downward in the atmosphere. However, Figure 3 shows that the relative contribution of different heights is practically the same for the whole region where the series expansion is useful; the really interesting region, where the lower atmosphere dominates the refraction, is beyond the reach of this approach.

A more direct explanation would be to consider the refraction integrand in its original form, $(d\mu/\mu)\tan\zeta$, where μ is the refractive index, in Newcomb's notation. The refractive invariant makes $\tan\zeta$ largest at the bottom of the atmosphere—this is essentially Fabritius's (1878) argument—and of course the refractivity gradient is largest there too. Again we see that the refraction comes mainly from the lowest layers, and that Newcomb was confused.

Considering that the works of Bessel (1823), Biot (1836), Fabritius (1878), von Oppolzer (1901), and others that explained the matter clearly were already available, Newcomb's widely quoted remark that "there is, perhaps, no branch of practical astronomy on which so much has been written as on this and which is still in so unsatisfactory a state" seems to refer more to his own limited understanding than to the actual state of the subject. Perhaps he was misled by Bruhns (1861), who overlooks Ivory's warning that the series is only semiconvergent, fails to emphasize the importance of Oriani's theorem, and dismisses Biot's important work as merely a "mechanical quadrature." (Mahan [1962] seems to have been similarly misled by Bruhns.)

8.3. *O'Connell*

O'Connell's (1958, 1961) argument is based on the observation that terrestrial objects appear at the horizon undistorted while silhouetted against the distorted image of the Sun. From this observation, he draws the erroneous conclusion that the distortion must arise in air well beyond the undistorted objects, that is, in the upper atmosphere.

The situation can be correctly understood by applying the method used by Wegener (1918). Suppose we divide the atmosphere into two parts, one above eye level and one below it. The upper part contributes most of the refraction, but its contribution near the horizon is nearly constant and hence cannot account for the complex fine structure observed in the setting Sun. The lower part is primarily responsible for the phenomena seen by O'Connell within 37′ of the visible horizon, because the dip of the horizon as seen from the 450 m height of the Vatican Observatory is 37′ for the Standard Atmosphere.

Now consider this lower part as a refractive optical element. The horizon ray has its lowest point at the visible horizon; rays a little above the visible horizon also have their minimum heights near the horizon. But, in a spherically stratified atmosphere, each ray is symmetric about its lowest point. So if we regard this lower part as a thick lens, its principal planes must also be symmetrically placed before and behind the horizon. Objects at the horizon are therefore situated *between* the principal planes of this atmospheric lens.

As the height of a ray at one principal plane is also the height of the ray at the other, it follows that the ray *heights* of objects near the horizon are practically unaffected by the atmospheric structure there; indeed, this should have been obvious from the very fact that the rays are nearly horizontal in this region. So ships at the horizon appear nearly undistorted. But, as this is the region in which rapid changes in ray *directions* occur, it is also the region responsible for the distorted images of objects optically at infinity (e.g., the Sun).

That O'Connell's error could stand unchallenged for 45 years shows a great ignorance of optics among both astronomers and meteorologists. T. S. Jacobsen (1959), reviewing O'Connell's book in PASP, declared the "scintillation" evidence "complete enough to show separately results for the sea horizon and for the solar limb, thus indicating whether the disturbances are relatively near or far away"; the reviewer for the Quarterly Journal of the Royal Meteorological Society (Oddie 1961) thought "that the distortion is mostly due to refractions in the upper air is demonstrated" by the famous ship silhouette.

For readers who do not understand thick lenses, I can only offer the example of the distorting glass often used in shower-stall doors and bathroom windows: you can read a newspaper through it if the paper is in contact with the glass, but objects farther behind the distorting medium are unrecognizable. The lower atmosphere acts in a similar fashion.

Another way to understand low-Sun distortions is to regard them as mirages. As Biot (1810) proves in his monograph, a mirage (an inverted image) can only occur if rays from the upper and lower portions of the miraged object cross between the eye and the object, so that the image of the upper part of the object is seen below the image of the lower part. The refractive invariant requires that rays intersecting at the observer's eye must diverge *above* the observer, so celestial mirages require the rays to cross somewhere *below* eye level, and the rays must be horizontal somewhere in this region. As the horizontal part of a ray is near the apparent horizon, only objects *beyond* the horizon can be seen miraged, in a horizontally stratified atmosphere. The miraging increases with distance beyond the horizon, so the most distant objects (e.g., the Sun) suffer the greatest distortions.

O'Connell's error is compounded by inappropriately applying the term "scintillation" to the image distortions. Taken properly, scintillation refers to rapid fluctuations of stellar irradiance produced by turbulence. And, as Wegener (1928) says in his encyclopedia article cited by O'Connell (1958), scintillation is produced mostly by the upper atmosphere. But what O'Connell calls "scintillation" is just complicated refraction due to the horizontally stratified atmosphere, not turbulence. Indeed, as Stanley P. Wyatt (1959) commented on the photographs in O'Connell's book, "Perhaps the most impressive feature of these weird shapes is their symmetry around a vertical axis, indicating that small-scale stratification in the atmosphere occurs in essentially horizontal layers." The layers below eye level produce the most striking effects, and the 450 m height of the Vatican Observatory allowed room for a multitude of thermal inversions below it, each producing a characteristic refraction feature of the mock-mirage type (Young et al. 1997).

8.4. *Kolchinskii*

The suggestion by Kolchinskii (1967), that the lack of proportionality between refraction and refractivity near the horizon has something to do with the upper atmosphere, is a complete non sequitur. Approximate proportionality holds near the zenith, where the small-angle approximation to the sines in Snel's law is good. Near the horizon, the lack of proportionality is due to the nonlinearity of the sine function—or rather, its inverse—as is easily seen from the exact formula for the plane-parallel atmosphere:

$$r = \arcsin(n \sin z) - z . \tag{3}$$

In this model the refraction is independent of the structure of the atmosphere; yet, at 88° Z.D. the refraction for $n = 1.00027$ is only 0.8826 as large as that for $n = 1.00030$, although the refractivity ratio is exactly 0.9000. Clearly, the failure of proportionality is due to the nonlinearity of Snel's law, not to atmospheric structure. The same effect applies in the real, curved atmosphere, though it is not so easily demonstrated.

However, the underlying error, which is to suppose that refraction near the horizon *should* be proportional to atmospheric refractivity at the observer, has been committed repeatedly in the astronomical literature—especially in the part of it dealing with green flashes. An incomplete list of examples includes Ranyard (1889), Henry (1891), Bauschinger (1896), Whitmell (1897), Julius (1901), Rambaut (1906), Braak (1915), Guillaume (1919), the younger Lord Rayleigh (1930), and Hoppe (1941). Apparently (Ivory 1823), this erroneous assumption goes back at least to the tables published by the French Board of Longitude in 1806. That it is incorrect was pointed out by Brinkley (1818), as well as by Ivory (1823), but by the end of the 19th century their warnings had been forgotten.

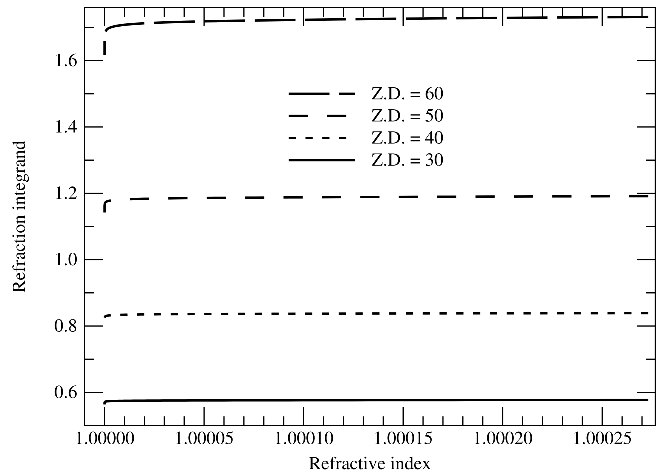

FIG. 11.—Integrand of eq. (4) for zenith distances of 30° to 60°. The observer is at the right side of the figure; the top of the atmosphere (where $n = 1$) is at the left. The calculation is for the Standard Atmosphere; the curves end at 120 km height.

9. UNDERSTANDING REFRACTION

Obviously, the standard treatments of refraction have not clarified its physics enough to prevent people such as Ivory, Newcomb, and O'Connell from falling into serious errors. Most discussions have focused on the details needed to construct refraction tables: atmospheric models, series expansions, and the evaluation of integrals. These details have distracted attention from the basic principles needed to understand the refraction problem.

Astronomical treatments emphasize the part of the sky where positional astronomy is usually done. This is the region where Oriani's theorem applies, so atmospheric structure is of minor importance. On the other hand, low-altitude refraction depends mainly on the lapse rate near the observer—compare Biot's theorem—with a weaker dependence on the rest of the lower troposphere. This zone has been largely neglected in theoretical discussions of astronomical refraction. The difference between these two domains was pointed out by Kurzyńska (1987), who noted the transition near 85° Z.D. but did not explain it.

The basic difference between the two regions is evident if we write the refraction integral as

$$r = \int_1^{n_o} \tan \zeta \frac{dn}{n} . \tag{4}$$

As n_o differs from 1 by less than 0.0003, the n in the denominator is hardly important; the interesting effects all come from $\tan \zeta$. Let us consider some particular cases.

9.1. *Small Zenith Distances*

Near the zenith, $\tan \zeta$ is nearly constant in the whole atmosphere, and nearly equal to $\tan z$, where z is the observed zenith distance. Figure 11 shows the integrand of equation (4) for several small zenith distances. To a good first approximation, a mean value of $\tan \zeta$ can be factored out of the integral. Also, because of the tiny range of n, we can replace the n in the denominator by a mean value of n—say, $(n_o + 1)/2$—and take it

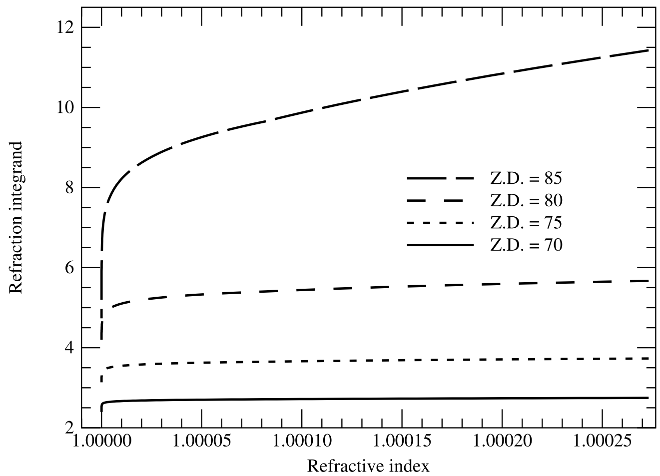

FIG. 12.—Integrand of eq. (4) for zenith distances of 70° to 85°. Cf. Fig. 11.

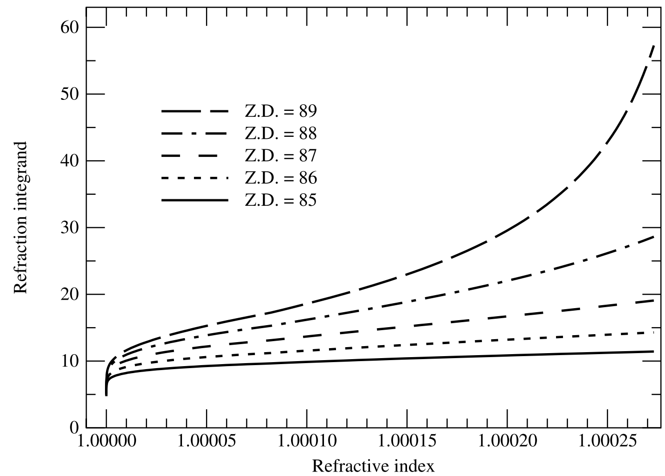

FIG. 13.—Integrand of eq. (4) for zenith distances of 85° to 89°. Cf. Fig. 11.

outside the integral. This leaves only dn in the integrand, so the integral evaluates to $n_o - 1$.

If Earth were flat, the mean value of tan ζ would be slightly greater than tan z. However, in a curved atmosphere, the mean value is smaller. The weighting by dn is equivalent to weighting by atmospheric density: $\rho \propto n - 1$ means that $d\rho \propto dn$. So a good mean value of tan ζ is the value at the density-weighted height, which is the height of the homogeneous atmosphere, h. From elementary trigonometry, this is $R/(R + h)$ times tan z at the observer. Then the near-zenith approximation becomes

$$r = \frac{R}{R+h}\,\frac{2(n_o - 1)}{n_o + 1}\,\tan z_o \tag{5}$$

or

$$r = \frac{2}{1 + h/R}\,\frac{n_o - 1}{n_o + 1}\,\tan z_o. \tag{6}$$

As is evident from Figure 11, the slight decrease of tan ζ with height in the atmosphere produces a slight slope in the integrand, and a slight rounding of the corner at the top of the atmosphere. These small effects are allowed for, to first order, by the second term in the traditional series expansion; Oriani's theorem shows that they are independent of atmospheric structure. This independence corresponds to the lack of any visible feature in the figure at the height of the tropopause, where n is about 1.00008.

9.2. *Oriani*

Oriani's theorem is due to the combined effects of thinness of the atmosphere and smallness of the refractivity. The thinness makes tan ζ nearly constant through the atmosphere out to considerable zenith distances, which accounts for the flatness of the integrand plots in Figure 11. Well away from the zenith, where the simple tangent approximation of equation (6) is not sufficient, the small refractivity makes the ray path nearly independent of atmospheric structure. This explains why the tropopause is invisible, even at 60° Z.D.

If we could set n to unity in the refractive invariant, it would reduce to R sin ζ = const. Then sin ζ (and hence tan ζ) would be a function of R alone, independent of atmospheric structure. But if n were 1, the ray would be straight—a close approximation, except near the horizon. Then the local zenith distance ζ would depend only on height, and not on the density distribution of the atmosphere.

We could then treat the refraction as a perturbation from this straight unrefracted ray. The (small) bending of the ray would depend only on the integrated change in n—which is proportional to the integrated change in density—along the ray. This is the physical basis of Oriani's result.

9.3. *Larger Zenith Distances*

Between 80° and 85° Z.D., the integrand develops appreciable slope (see Fig. 12), and Oriani's approximation breaks down. The range of ζ in the atmosphere is large enough to produce a general slope, and a markedly rounded corner at the top of the atmosphere. The visibility of the tropopause as a change in slope near $n = 1.00008$ shows that atmospheric structure affects the refraction; ray curvature modifies the local value of tan ζ appreciably, However, as the integrand is still only a *little* larger near the ground than in the stratosphere, refraction variations up to 80° or 85° Z.D. depend primarily on the average tropospheric lapse rate, if we change the atmospheric model.

This zone of zenith distance is where the traditional series expansion starts to fail. If the series converges well enough to be useful, its structure-dependent terms must be much smaller than the first two, which are structure independent. Therefore, in the whole region where the series expansion can be used, the refraction must be *nearly* independent of atmospheric structure. Only where the series expansion breaks down can the structure become important—that is, within a few degrees of the horizon. Thus the series-expansion approach is useless for understanding refraction at low altitudes.

The increasing slope of the integrand with zenith distance in Figure 12 shows the increasing importance of the lower atmosphere in this region. If anyone had plotted this function, the predominance of the lower atmosphere at lower altitudes would have been obvious, and the errors discussed above could have been avoided. But excessive emphasis on series expansions has apparently obscured the physics of refraction.

9.4. *Horizon Region*

Finally, Figure 13 shows the behavior of the integrand within 5° of the horizon. The convergence of the curves toward the left side reflects the near-constancy of the local zenith distance at the top of the atmosphere, as pointed out by

Biot (1836) and Fabritius (1878). On the right side, tan ζ diverges at the observer's horizon, where transformation of the refraction integral to the BAS form is required. The lowest layers, where ray curvature is very important, evidently dominate near the horizon.

As ray curvature depends on the lapse rate in the boundary layer, which varies with the weather, it is hopeless to predict refraction accurately close to the horizon; it is simply too variable. And it is even more hopeless to try to predict refraction here from the air density at the observer, as the variations in refraction due to varying density *gradients* far exceed those due to varying local density. Likewise, standard tables are useless near the horizon, except to check numerical integrations for accuracy.

Refraction near the horizon depends primarily on boundary-layer structure. It is practically uncoupled from the refraction between 70° and 80° Z.D., because the boundary layer that dominates refraction below 5° altitude contributes relatively little to the refraction integral above 10°, where atmospheric layers are nearly equally weighted by density.

Because astronomical observations are rarely made so near the horizon, it is useful to ignore this zone and consider a simple model in which the ray can be approximated by a straight line.

9.5. *Cassini*

For example, consider the homogeneous-atmosphere model, used by Cassini (1662). As Ivory (1823, p. 410) says, "Perhaps it is owing to its great simplicity, that the method of Cassini seems not to have met from astronomers with the attention it deserves." Ivory's comment remains true today; the only textbook I know that even gives Cassini's formula is that of Alekseev et al. (1983, p. 58), although it is well known that Cassini's model provides an easy way (cf. Ball 1915, pp. 125–127) to derive the first two terms in the series expansion. But suppose we adopt his model *exactly*, instead of making such approximations.

Figure 14 shows Cassini's model. The homogeneous atmosphere has a refractive index n and height h. The ray whose apparent zenith distance at the observer is z meets the top of the atmosphere at angle of incidence ζ. In the triangle POC, the sine of the angle at O is the same as the sine of its supplement, which is the observed zenith distance z. Then the law of sines gives

$$\frac{\sin\alpha}{R} = \frac{\sin z}{R+h} \tag{7}$$

(where just R is now used for the radius of Earth); hence,

$$\sin\alpha = \frac{R\sin z}{R+h}. \tag{8}$$

The law of refraction at P is

$$n\sin\alpha = \sin\zeta, \tag{9}$$

but the angle $\zeta = \alpha + r$, where r is the refraction, so

$$r = \zeta - \alpha = \arcsin\left(\frac{nR\sin z}{R+h}\right) - \arcsin\left(\frac{R\sin z}{R+h}\right). \tag{10}$$

Note that this result is exact; no approximations have been made.

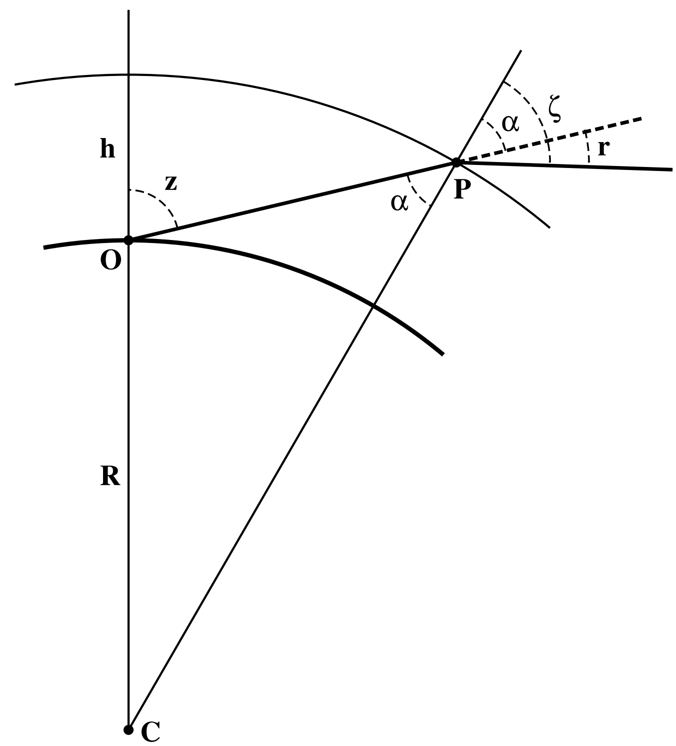

FIG. 14.—Cassini's homogeneous-atmosphere model. The heavy arc represents Earth's surface, with center C; the light arc is the top of the atmosphere. The observer is at O, where the apparent zenith distance is z; the ray is refracted at P, where the local zenith distance above the air is ζ.

Cassini adjusted the height h to reproduce the measured horizontal refraction, but as his model has an unrealistic density gradient (namely, zero) at the surface, we know from Biot's theorem that it must give unrealistic refractions at low altitudes. So Cassini's *table* distributed the large error at the horizon over the whole sky and was unreliable everywhere; perhaps this is why his *model* fell into such neglect. For numerical calculations, one should use the actual value of h derived from the surface pressure and density and forget about the horizon region. Doing this gives excellent values near the zenith, where the surface gradient is unimportant. However, Cassini's model then gives a horizontal refraction that is only 60% as large as that in the standard model.

10. COMPARISON OF MODELS

Figure 15 shows the differences between the refraction in the Standard Atmosphere, integrated numerically to high accuracy; the exact Cassini model; and refractions calculated by Stone's "accurate method" (Stone 1996), which is a truncated series expansion taken from the textbook by Green (1985). Refractions for the model NBL7 with a rather well-developed nocturnal inversion, and for isothermal and adiabatic atmospheres, are also shown for comparison.

Remarkably, Cassini's physically unrealistic model is more accurate than Stone's "accurate method" at *all* zenith distances. Why? Because Cassini's homogeneous model is still an atmosphere in hydrostatic equilibrium, while the truncated series does not correspond to even this minimal physical condition. However, the lapse rate required to offset the effect of decreasing pressure with height is some 34 K km^{-1} (Wegener 1918), more than 3 times the adiabatic lapse rate, so Cassini's homogeneous atmosphere would be dynamically unstable.

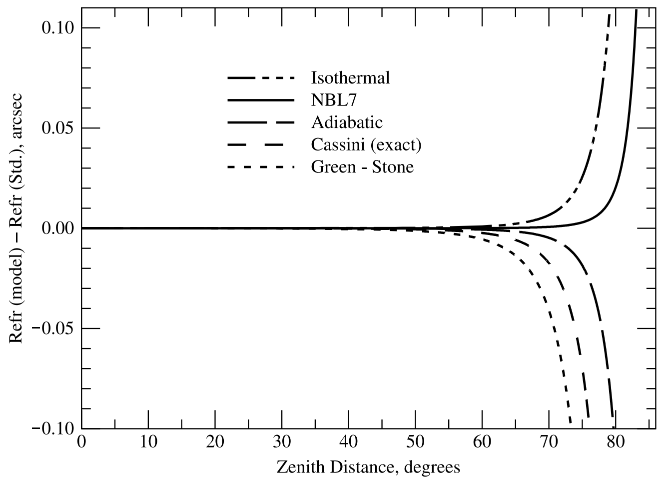

FIG. 15.—Refraction for various models *minus* refraction for the Standard Atmosphere, as functions of zenith distance. All models have the same temperature and pressure at the observer. The Standard Atmosphere is not shown; it would be a horizontal line at ordinate zero.

Of course Oriani's theorem ensures that Cassini's model is good near the zenith, but this homogeneous model continues to work well surprisingly far from the zenith. Cassini's model at least gives finite (if wrong) values of the right sign at the horizon, while the series expansion diverges several degrees above the horizon, and the truncated series used by Stone (1996) gives negative values below about 2° altitude.

In fact, at 70° Z.D. Cassini's model differs from the Standard Atmosphere refraction by only 17 milliarcseconds (mas), while the Green-Stone approximation is off by 41 mas. At 74° Z.D., Cassini is off by 51 mas and the Green-Stone approximation is off by 122 mas. In this region, the Cassini model has only 42% the error of Stone's "accurate" formula.

The superiority of Cassini's model to the truncated series expansion can be understood as follows: Truncating the series, whose terms' coefficients are essentially height moments of the refractivity, is equivalent to setting the omitted moments to zero. But that would require *negative* densities in parts of the upper atmosphere. Clearly, Cassini's model, while not very realistic, is less absurd than negative densities.

But, as Cassini's model itself is not realistic, one should be able to do still better. As refractions for different models differ appreciably beyond 74° Z.D., we need to ask what *range* of models can be considered realistic, and how much they differ.

10.1. *Range of Plausible Models*

The "adiabatic" atmosphere model shown in Figure 15 has a lapse rate of 10 K km^{-1} throughout. The real atmosphere cannot have such a steep lapse rate—the actual dry adiabatic lapse rate is slightly smaller than 10 K per kilometer—so this is surely a lower limit to the refraction. Likewise, the real atmosphere is never isothermal; so, that model is surely an upper limit to the range of plausible refractions, except within a degree or two of the horizon, where shallow inversions can produce much larger values.

Therefore, except very near the horizon, the actual refraction must be confined between these two models, which differ by only 11 mas at 70° Z.D., and 44 mas at 75°. Figure 15 shows that the Standard Atmosphere is about midway between these two extreme models, which helps explain why Standard Atmosphere refraction has proved to be so satisfactory in practice. Even at 77° Z.D., the isothermal model (which is slightly worse than the "adiabatic" model) differs from the standard model by slightly less than 50 mas, the tolerance limit set by Stone (1996) for accurate work. So refraction based on the Standard Atmosphere is certainly not in error by as much as 50 mas even at 77° Z.D., and perhaps a bit farther in most cases.

This limit is considerably larger than a literal reading of Oriani's mathematical result would suggest. But his theorem applies to *all possible distributions* of refractivity, regardless of whether they are stable against convection, or are probable states of the atmosphere. That is, Oriani's theorem would still apply if the atmosphere were replaced by an equivalent mass of water, or if the density of the atmosphere increased upward instead of decreasing. Physically possible atmospheres are a small subset of the refractive structures to which the theorem applies, and *likely* states of the atmosphere are an even smaller set. (Note that Biot [1838, 1854] made similar arguments long ago: the possible states of the atmosphere are sufficiently limited that the range of possible refractions is likewise limited.)

10.2. *Comments on the Isothermal Model*

The isothermal model differs from the standard model by less than 0.4 as much as does Cassini's model, so it is even better suited than Cassini's for use as an approximate refraction. Its error, compared with the standard model, is only 26 mas at 75° Z.D., 49 mas at 77°, and 105 mas at 79°—about 7 times smaller than the errors of Green's truncated series used by Stone (1996).

However, the isothermal model has long been known to give too large a refraction near the horizon. It was first used by Newton; but even in his day, Flamsteed could see that observations of refraction did not fit the model—a discrepancy that hastened the falling-out between Newton and Flamsteed.

The isothermal model's deficiencies have sometimes been blamed on its too-warm upper regions. But, in view of the systematic behavior shown in Figures 7 and 9, the real reason for the excessive *horizontal* refraction of this model is obvious: it has the wrong lapse rate at the observer. Indeed, even Brook Taylor (1715) found, in the last paragraph of his Methodus Incrementorum (see Feigenbaum 1981 for an English translation), that the radius of curvature of a horizontal ray in the exponential model is about 5 times the radius of Earth; but the true ratio is closer to 7, the value adopted by Lambert (1759) from an analysis of Cassini's measurement of dip of the sea horizon, and well established in geodesy since then. This excessive curvature of the isothermal model's horizontal ray keeps it near the ground longer than is realistic, thus producing the excessive refraction.

However, the lapse rate at the surface is not the whole story. The NBL7 model differs even more from the standard lapse rate than does the isothermal model, but it lies between the isothermal model and the standard in Figure 15. Yet, as is well known, the isothermal model differs from the standard model by only 3.′2 at the horizon, while NBL7 produces 11.′3 larger horizontal refraction than standard does. In fact, these two models cross 41′ above the horizon, where they both exceed the standard-model refraction by 89″.

Similarly, the NBL9 model, with twice as big an inversion as NBL7, differs nearly twice as much from the standard-model refraction near 75° Z.D. as the NBL7 model does, but it still crosses the isothermal model at an altitude of 1°25′, where they both exceed the standard refraction by nearly 43″. Evidently, as was already seen in Figure 12, the refraction between 75° and 80° Z.D. depends on the mean lapse rate of the whole atmosphere, but the effect is small, hardly amounting to a tenth of an arcsecond up to 79° or 80°.

It should be clear from these examples that the differences in refraction between models with the same conditions at the observer represent a smoothed picture of their temperature profiles: differences in the upper parts of the model atmospheres produce differences in refraction 10° or 15° above the horizon—where, in any case, the differences among physically plausible models are a fraction of a second—while differences in the boundary layer, just above the observer, produce much larger differences in the horizontal refractions, amounting to many minutes, or even a few degrees, of arc.

11. CONCLUSION

If the refraction integrand is kept in the form

$$\tan \zeta \frac{dn}{n} \tag{11}$$

instead of being transformed, it is clear where the refraction comes from (see Figs. 11–13). The refractive index n varies only between 1.0000 and 1.0003, while $\tan \zeta$ varies from zero to infinity; obviously, $\tan \zeta$ is responsible for all the important effects.

Physically, the ray slope $\tan \zeta$ depends on its curvature only near the horizon. The ray curvature is proportional to the component of the refractivity gradient perpendicular to the ray, which is very small near the zenith and largest at the horizon. So the local zenith distance ζ is nearly constant along the ray near the zenith, which accounts for Oriani's theorem: atmospheric structure is unimportant here. But, by the argument of Fabritius (1878), ζ varies strongly near the horizon, because of the refractive invariant. Consequently, $\tan \zeta$ is enormous near the ground for rays near the horizon, and *much* smaller a few kilometers higher up: the lowest layers become progressively more important near the horizon, where the lapse rate at the observer plays a dominant role, as it determines the ray curvature where $\tan \zeta$ is largest.

Unfortunately, this straightforward physics is completely obscured by the transformations of variable used to construct refraction tables by the series-expansion method introduced by Lambert (1759). Although Ivory (1823) clearly shows that many terms must be used in the series expansion, even near the zenith, because the series is only semiconvergent, textbooks such as Ball (1915) and Green (1985) have given the impression that truncating the series after the second term provides an adequate approximation—even though it is worse *everywhere* than Cassini's model.

The series-expansion approach is misleading, inaccurate, and inefficient compared with the numerical quadrature method introduced by Biot in 1836, which is now the recommended method (Seidelmann 1992) for calculating refraction. It is time for series expansions in $\tan z$ to disappear from the textbooks and fade into history. If a simple approximation for refraction is needed, Cassini's model is quite adequate for most practical purposes out to about 80° Z.D.

Refraction within about 5° of the horizon is so variable that no a priori formula or table can be expected to give accurate values there; the local lapse rate and thickness of the boundary layer above the observer must be known. However, numerical integrations using the standard lapse rate, matched to the actual temperature and pressure at the observer, should give values good to a minute of arc or so—that is, good enough for telescope pointing—down to 2° or 3° apparent altitude.

Below that, numerical integrations will give useful values if the actual boundary-layer lapse rate and thickness are known, but only the bottom kilometer or so of the atmosphere needs to be measured, and ordinary balloon soundings are adequate. At and below the astronomical horizon, the refraction depends primarily on atmospheric structure *below* the observer and varies so much (tens of minutes, or even several degrees) that only very crude predictions can be made. The observed time of sunset at a sea horizon often varies by a few minutes from day to day, and the variations increase with height above the sea.

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